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ORIGINAL ARTICLE

A Colorful Approach to Knot Theory (or: "How Happy I Could Be With Ether")

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Abstract

Ralph Fox's p-colorings offer the simplest yet effective invariants of knots. This exposition, based on the 2020 Lewis-Parker Lecture delivered by the author, introduces the method of knot coloring as well as some historical background.

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1. "My soul is an entangled knot..." – physicist James Clerk Maxwell

Knot theory is an active area of mathematical research. Nevertheless its main object of study and its primary goal can be described with only a handful of words. Think of a knot as an elastic cord tangled up with its ends connected. Given two knots, decide whether they are the same, that is, whether one can be deformed to appear identical to the other.

We can prove that two knots are the same by performing a clever deformation. But how can we tell that two knots are not the same? We cannot physically test every possible deformation.

As an example, consider the three knots depicted in Figure 1. Two of them are the same. Can you decide which two?

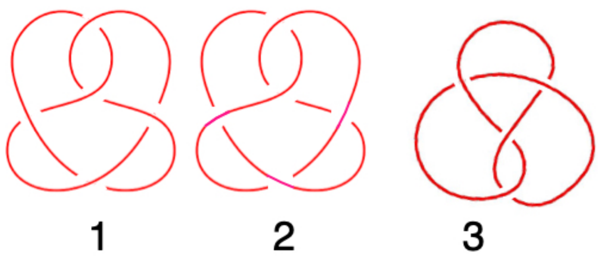

Figure 1. Which pair of knots are the same?

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Knots 2 and 3 are in fact the same, something that the diligent reader can verify with a piece of string. However, showing that knot 1 is different requires a mathematical idea. We present a simple, colorful idea that can be used to show that knot 1 is the odd knot out. It can also be used to distinguish many other pairs of knots. It is the method of knot colorings.

Knot colorings were the idea of the Princeton topologist Ralph Fox, who introduced them in 1956. To apply them we need only elementary ideas of linear algebra.

Figure 2. Ralph Fox at the celebration of his 60th birthday. (The autographed photograph was a gift to the author by Wilbur Whitten.)

2. A wee bit o' knot theory and its history

We are taking an informal approach to knot theory. The reader can find more details in one of the many excellent books on the subject (e.g. [2–4]).

A *knot* is a smoothly embedded circle in $\mathbb{R}^3$. Two knots are *equivalent* if one can be smoothly deformed to be identical with the other. Such a deformation allows bending, stretching and shrinking, for example, but not breaking or pulling so tightly that a singularity is created. We regard equivalent knots as the same.

While knots attracted the interest of 18th and early 19th-century mathematicians such as Alexandre-Théophile Vandermonde and Carl Friedrich Gauss, they became the subject of sustained study only later when the Scottish physicist Peter Guthrie Tait began classifying them. Tait's inspiration came from Lord Kelvin's "vortex atom theory," a theory that atoms are small knots of an invisible, frictionless fluid called *ether*, a substance believed to permeate all of space. Tait began an ambitious program of knot tabulation in 1876 with the expectation that he was classifying the chemical elements. We call it "ambitious" because he had few topological theorems or techniques to aid him. What Tait did have was marvelous mathematical intuition.

Tait interested his life-long friend James Clerk Maxwell in the study of knots. Maxwell, one of the greatest scientific minds of the 19th century, had both a sense of play and a sense of humor. He was also something of a poet. Playing with lyrics to John Gay's *The Beggar's Opera*, Maxwell wrote to Tait: "How happy I could be with ether!"

Unfortunately for Lord Kelvin, no ether was ever found. The vortex atom theory died, but modern knot theory had been born.

Figure 3. Peter Guthrie Tait (left); James Clerk Maxwell with wife and dog (right)

3. "Color is the place where our brain and the universe meet." – artist Paul Klee

The easiest way to describe a knot is to present a type of picture called a *knot diagram.* It is a projection D of the knot in the plane seen as a 4-regular graph with a hidden line effect in a neighborhood of each vertex indicating how one part of the knot passes over another. (Figure 1 contains three examples of knot diagrams.) A maximal connected subset of D is called an *arc* of the diagram. We leave it as an exercise to verify that the number of arcs of D is the same as the number of crossings.

Figure 4 shows some local changes of any knot diagram that do not change the knot. By a "local change" we mean a change that leaves points outside of the picture fixed. The 0th change should be regarded as any smooth deformation that does not introduce or destroy any crossings. (Such changes typically involve bending, stretching or shrinking.)

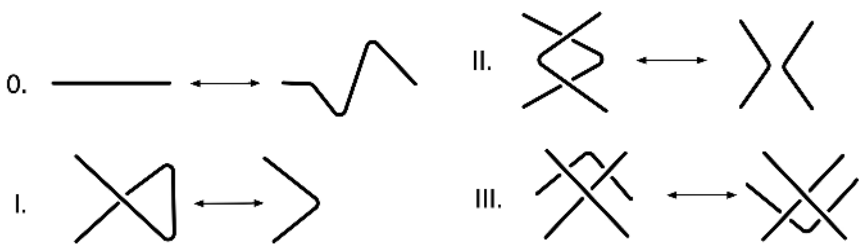

Figure 4. Reidemeister moves

The local changes in Figure 4 are called *Reidemeister moves.* They are named after Kurt Reidemeister, who proved in 1927 that if two diagrams represent the same knot, then some finite sequence of the moves will convert one diagram into the other. Figure 5 illustrates an application of a Reidemeister move of type (ii) followed by a move of type (i).

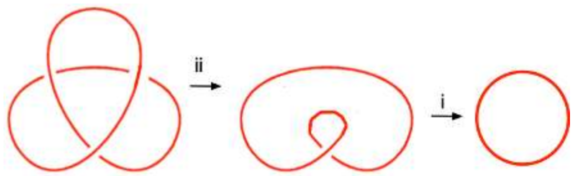

Figure 5. Reidemeister move of type (ii) followed by a move of type (i)

Reidemeister's theorem is remarkable for at least two reasons. First, the theorem shows us that knot theory can be approached as combinatorics by studying knot diagrams and Reidemeister moves. Secondly, it implies that quantities (e.g., numbers, vector spaces, polynomials) associated to a knot diagram that are unchanged by Reidemeister moves are in fact invariants of the knot. If such a quantity takes different values on two knot diagrams, then the knots that the diagrams represent are different. QED!

By coloring knot diagrams, we will find some of the simplest yet effective knot invariants.

Let $\mathbb{Z}/p$ denote the field of integers modulo prime p. Elements are $0, 1, 2, \ldots, p-1$. (Remember that $a+b$ is the remainder upon division by p.) Knot theorists like to call $0, 1, 2, \ldots, p-1$ *colors*. There is no mathematical need to do this. However, thinking about elements of $\mathbb{Z}/p$ as colors enlivens our imagination.

Figure 6. Visualizing mod-3 colors

A *p-coloring* of a knot diagram D is a labeling of the arcs by colors $a, b, c, \ldots$ so that at any crossing the *p-coloring relation* is satisfied: $2a - b - c = 0$ modulo p, where a is the color of the over-crossing arc and b, c are the colors of the other two arcs. (For other approaches to knot coloring see [1].)

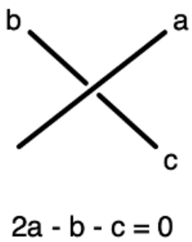

Figure 7. p-coloring relation

Any knot diagram can be p-colored by assigning the same color to every arc. Such monochromatic p-colorings are uninteresting. We will eliminate all of them with the exception of the *trivial p-coloring*, the p-coloring that assigns 0 to every arc, by selecting a *base arc* and requiring that it receive the color 0.

When $p = 3$, our p-coloring relation is pleasingly simple. It can be restated by the requirement that at every crossing all colors are different or else they are the same. A few examples appear in Figure 8.

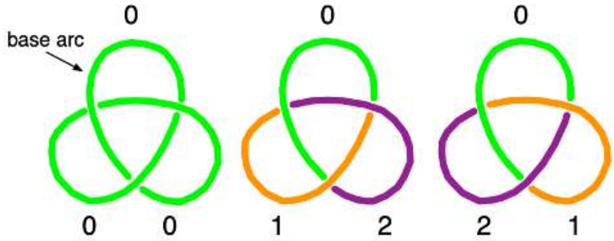

Figure 8. Three 3-colorings of a knot diagram

Since the coloring relation is linear, we can add two p-colorings arc-wise to get another p-coloring. For example:

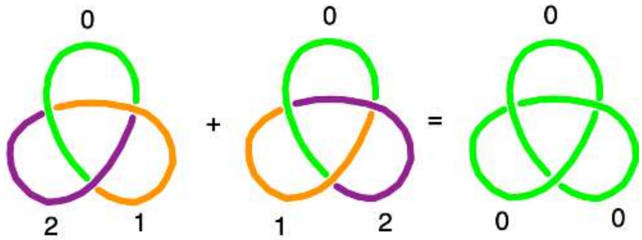

Figure 9. Adding 3-colorings

We can also multiply p-coloring by elements of $\mathbb{Z}/p$. In fact the p-colorings of a diagram D form a vector space. We will denote it by $\mathrm{Col}_p(D)$ and call it the *Fox p-coloring space* of D. The reader is encouraged to prove that the dimension of the space does not depend on our choice of base arc.

By now you might have guessed that for any prime p the Fox p-coloring space is a knot invariant. Proving it is surprisingly easy. We will present part of the proof, leaving the rest as an exercise.

Theorem 3.1. *If D and D' are diagrams of the same knot, then* $\mathrm{Col}_p(D)$ *and* $\mathrm{Col}_p(D')$ *are isomorphic, for any prime p.*

Proof. It suffices to assume that D' is obtained from D by a single Reidemeister move. We consider Reidemeister move (ii), and check that c is uniquely determined by a and b, and also that $d = a$.

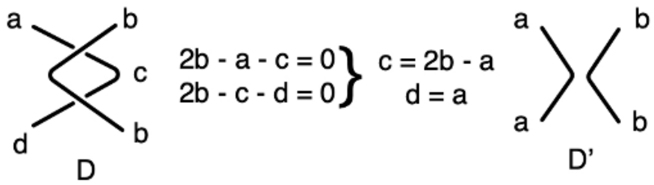

Figure 10. The color c is uniquely determined by colors a and b.

Any p-coloring of D corresponds to a p-coloring of D', and conversely. Moreover, the correspondence respects addition and scalar multiplication. Reidemeister moves of types (i) and (iii) are handled similarly. □

If D is a diagram of a knot k, then by Theorem 3.1 we can write $\mathrm{Col}_p(k)$ instead of $\mathrm{Col}_p(D)$. Any other diagram for k would produce an isomorphic vector space. We will call $\mathrm{Col}_p(k)$ the *Fox p-coloring space* of k.

4. "If two people always agree, one of them is redundant." – Ben Bernanke, former Chair of the Federal Reserve

When we p-color a knot diagram D, we discover that not all of the relations at the crossings are needed. As Lemma 4.1 below asserts, any single relation is redundant. This non-trivial fact is a consequence of the topology of the plane. Most proofs involve a good dose of algebraic topology. However, the proof that we present is elementary. Also, it gives a taste of low-dimensional topology techniques. Those in a hurry are welcome to skip over the proof during their first reading.

Lemma 4.1. *Let D be a knot diagram and p a prime. Then any p-coloring relation is a linear combination of the remaining p-coloring relations of D.*

Proof. If the knot diagram D has no crossings, then there is nothing to prove. Therefore we assume that D has at least one crossing.

Consider what happens when we follow the path of a small circle around any crossing and record the colors that we see as an alternating sum, first positive then negative. If the first arc we meet is an over-crossing, then the sum we get is the p-coloring relation of the crossing. On the other hand, if the first arc is not an over-crossing, then we obtain the p-coloring relation multiplied by -1. An example is given in Figure 11.

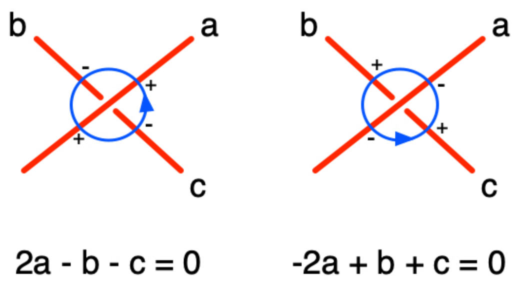

Figure 11. Reading the relation around a crossing (starting at the arrowhead)

We place a small circle about every crossing of D, remove small intervals from the circles in order to join the results together by narrow double arcs disjoint from D and create a single simple closed curve Γ, as in Figure 12. (There are many ways to do this.) Beginning at any point of Γ and traveling around it in either direction, we attach alternating signs $\pm$, beginning with $+$, to the points of $\Gamma \cap D$, as in Figure 12.

Let Σ be the alternating sum of the labels of the arcs that we cross, with signs corresponding to those of the points of $\Gamma \cap D$. Every circle contributes the p-coloring relation of the enclosed crossing or else the relation multiplied by -1. (Although the path around a circle might be interrupted as we travel along Γ, the detour meets D in a number of points that is even, in fact a multiple of 4, and hence the pattern of sign alternation around each circle is preserved.) We complete the proof by showing that the weighted sum Σ of p-coloring relations is zero.

The curve Γ separates the plane into two regions. One region is bounded, topologically a disk, and contains all the crossings of D. The other region is unbounded. We follow the knot diagram from any point in either direction. If an arc of D crosses Γ at a point P and enters the unbounded region, it must eventually return to the bounded region. When it returns, it crosses Γ at a point Q with the opposite sign of P. To see why, consider the set of all arc segments in the unbounded region. They cannot intersect each other since all the crossings of D are contained in the bounded region. Either P and Q are consecutive points of $\Gamma \cap D$ (as encountered as we travel along Γ) or else they are separated by other arc segments and hence by an even number of intersection points. In either case, P and Q have opposite signs, and so their combined contribution to Σ vanishes. Since all contributions to Σ arise from endpoints of the such arc segments, Σ is zero. □

Example 4.2. We illustrate the ideas of the proof of Lemma 4.1. Consider the diagram of Knot 2 as it appears in on the left-hand side of Figure 12. The closed curve Γ is blue. We compute Σ as we travel around Γ, beginning at the arrowhead. To see that the terms cancel in pairs, consider the disk that Γ bounds. The segments of the arcs of D that leave and reenter the disk are indicated on the right-hand side of Figure 12. Note that between the endpoints of any arc segment there are an even number of points. The signs of the endpoints of each arc segment are opposite, and so the terms of Σ cancel in pairs.

The vanishing sum Σ can be rewritten as $(2a - b) + (2b - a - c) - (2b - d) - (2a - c - d) - (2d - a - b)$, a linear combination of the p-coloring relations.

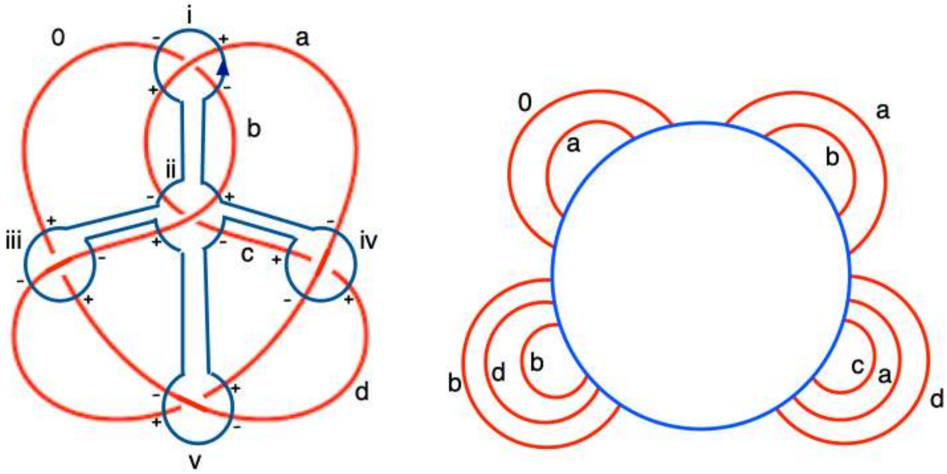

Figure 12. Diagram D and closed curve Γ (left); arc segments (right)

5. "Color is my daylong obsession, joy, and torment." – artist Claude Monet

Given a knot k and a prime p, we are now in position to determine the Fox p-coloring space $\mathrm{Col}_p(k)$. First, choose a diagram D for k. Then select a base arc, and label the other arcs by $a_1, \ldots, a_n$. (If there are only a few arcs, then we can avoid subscripts by using $a, b, c, \ldots$ instead.) Build an $n \times n$ matrix M with rows corresponding to the arcs of D and columns corresponding to any $n - 1$ of the p-coloring relations. Which relation we omit will be unimportant thanks to Lemma 4.1. We call M a *coloring matrix* of D.

Proposition 5.1. *Assume that D is a diagram of a knot k and p is a prime. If M is a coloring matrix of D, then $\mathrm{Col}_p(k)$ is a nontrivial vector space if and only if $|\mathrm{Det}(M)|$ is divisible by p.*

The proof of Proposition 5.1 follows immediately from what has already been said. We can conclude a bit more: the p-colorings of D correspond to the vectors in the column mod-p null space of M. Remember that the base arc always receives the color 0.

Example 5.2. Consider Knot 1 of Figure 1. With the arcs of its diagram labeled and crossings numbered as in Figure 13, we have

$$M = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & -1 & 0 & 2 \end{pmatrix}.$$

Since $|\mathrm{Det}(M)| = 7$, the diagram can be non-trivially p-colored only for $p = 7$. The reader can verify that $a = 5, b = 3, c = 4, d = 1$ is a 7-coloring of the diagram.

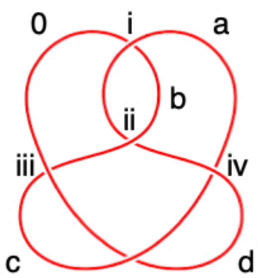

Figure 13. Knot 1

Consider Knot 2 of Figure 1 with diagram labeled as in Figure 14.

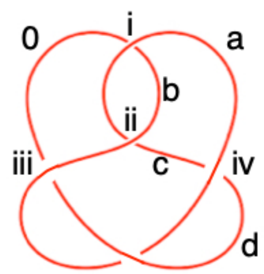

Figure 14. Knot 2

For this diagram we have:

$$M = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & 2 & 0 & -1 \\ 2 & 0 & -1 & -1 \end{pmatrix}.$$

Here $|\mathrm{Det}(M)| = 5$, and so the diagram can be non-trivially p-colored only for $p = 5$. The reader can verify that $a = 4, b = 3, c = 2, d = 1$ is a 5-coloring.

In view of Proposition 5.1, we have now shown that knots 1 and 2 are in fact different.

6. How Happy Tait Would Have Been!

For any knot k, the set of primes p for which a diagram of k can be non-trivially p-colored is an invariant of the knot. We will call it the *knot spectrum* of k.

Proposition 6.1. *The knot spectrum of any knot is finite.*

Proof. Suppose that the knot k has an infinite knot spectrum. If M is a coloring matrix for a diagram D of k, then $|\mathrm{Det}(M)|$ is divisible by an infinite number of primes, which implies that $\mathrm{Det}(M) = 0$. Since 2 divides $\mathrm{Det}(M)$, Proposition 5.1 implies that the diagram D can be non-trivially 2-colored. However, the only 2-colorings of a knot diagram are constant. Since the base arc of D receives the color zero, D can be 2-colored only trivially, contradicting our initial assumption. Hence the knot spectrum of k is finite. □

Knot spectra are reminiscent of the atomic spectral lines of an atom, the characteristic lines that elements display in the flame of a Bunsen burner. Had Tait known about p-colorings in the early, heady days of his investigation of the vortex atom theory, how happy he would have been! Undoubtedly, he would have seen it as more evidence of Lord Kelvin's theory that elements were knots of ether.

But analogies go only so far. While no two elements share the same atomic spectral lines, two knots can have the same knot spectrum. In fact, there are infinitely many different knots having an empty knot spectrum! (Fortunately, we have other invariants that can distinguish knots.)

We hope that the reader has been inspired to learn more about knot theory. It is a rich subject that has developed in ways that Tait and Fox could never had imagined. While the vortex atom theory has been forgotten, some modern knot theory techniques have now drawn back the attention of physicists. We recommend browsing through some recent issues of the journal *Knot Theory and its Ramifications* to get a feeling for some of these developments.

Knot theory is still young. After all, the subject has been around for little more than 150 years. Knots seem like toys. However, our instincts tell us that knots model some things in ways that we do not yet understand. Today knot theory is being used in studies of DNA, hydrodynamics, robotics and more, but it is possible that the most important applications are waiting to be discovered.

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