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Slope Conjecture and Normal Surface Theory

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THE UNIVERSITY OF SOUTH ALABAMA
COLLEGE OF ARTS AND SCIENCES

THE SLOPE CONJECTURE AND NORMAL SURFACE THEORY

BY

Helene Swanepoel

A Thesis

Submitted to the Graduate Faculty of
The University of South Alabama
in partial fulfillment of the
requirements for the degree of

Master of Science

in

Mathematics

May 2022

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ABSTRACT

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In this thesis we explore the relationship between quantum link invariants and the geometric and topological properties for a family of pretzel knots $P(-2r, 2r+1, 2r+s)$, $r \geq 1$ and $s \geq 3$, as predicted by the Slope conjecture. This conjecture asserts a connection between the degree of the colored Jones polynomial and boundary slopes of these knots. We introduce fundamentals of knot theory and prove that the minimum degree of the Jones polynomial for the family of pretzel knots $P(-2r, 2r+1, 2r+s)$, $r \geq 1$ and $s \geq 3$, is $-6r - 3r + 3$.

In addition, we consider normal surface theory to find the boundary slopes of essential surfaces in the knot complement. As example, we consider the Lens Space $L_{3,1}$ and compute the Q-matching equations associated with the triangulation. Since every essential surface is isotopic to a Haken sum of normal surfaces, we hope to one day combine the theorem proven in this thesis with normal surface theory in order to prove or disprove the Slope conjecture for the family of pretzel knots $P(-2r, 2r+1, 2r+s)$.

CHAPTER I

INTRODUCTION

One of the main goals of knot theory is to identify properties that distinguish knots. These properties are known as knot invariants. In 1984, Vaughan Jones discovered a new polynomial for knots and links called the Jones polynomial [1]. This was a major breakthrough as Jones polynomials were introduced as knot invariants.

In order to define the Jones polynomial, we use Kauffman's formulation of this polynomial called the bracket polynomial. We then use this polynomial, along with the Chebyshev polynomial, to calculate the Jones polynomial.

The Slope conjecture, by Garoufalidis, predicts a connection between the degree of the colored Jones polynomial and boundary slopes of knots [5]. This is a difficult problem since the definition of the color Jones polynomial is combinatorial and based on the diagram of the knot, which does not easily relate to the topology of surfaces lying inside of the three dimensional manifold that is the knot complement.

The Slope conjecture is known for alternating knots [5], adequate knots [3, 4], iterated torus knots [11], families of 3-tangle pretzel knots [12], knots with up to 9 crossings [5, 11], family of 2-fusion knots [6] and graph knots [16]. Lastly, knots obtained by iterated cabling and connect sums of knots from any of the above classes also satisfy this conjecture [16].

We first present the ideas involved in knot theory such as knot invariants. Consequently, fundamentals of knot theory are introduced in Chapter II. Here, we discuss knot invariants, the Kauffmann bracket, the Jones polynomial, colored Jones polynomial and techniques needed to prove our main theorem.

In Chapter III, we prove that the minimum degree of the Jones polynomial for the family of pretzel knots $P(-2r, 2r+1, 2r+s)$ for $r \geq 1$ and $s \geq 3$ is $-6r - 3s + 3$. We are interested in this family since the Slope conjecture has already been proven for alternating knots and the family is non-alternating.

Next, we relate the degree of the colored Jones Polynomial to the slopes of essential surfaces using the slope conjecture in Chapter IV.

Finally, in Chapter V, we summarize the progress provided in this thesis and discuss upcoming research. In the future, we hope to use the theorem, proven in Chapter III, along with normal surface theory to prove or disprove the Slope conjecture for the family of pretzel knots $P(-2r, 2r+1, 2r+s)$.

CHAPTER II

KNOTS AND KNOT INVARIANTS

2.1 Knots and Knot Diagrams

Knot theory investigates how a one-dimentional “string” can lie in three-dimensional space. We start by defining a knot and then a special class of knots, called pretzel knots.

Definition 2.1.1. [13] A *link* L of m components is a subset of $\mathbb{R}^3$, that consists of m disjoint, piecewise linear, simple closed curves. A link of one component is a *knot*.

An example of a knot:

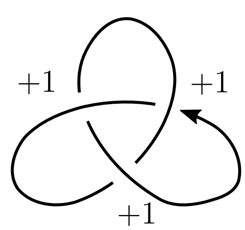


Figure 1. Trefoil knot

We use diagrams or projections to study knots. A *knot diagram* consists of edges and crossings [18]. Consider, for example, two different diagrams or projections of the trefoil knot:

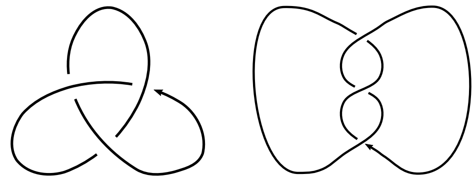


Figure 2. Trefoil knot and Braid trefoil knot

We assign an orientation to a knot diagram. An orientation of a knot diagram is an assignment of $\{+1, -1\}$ values to each crossing following the rule:

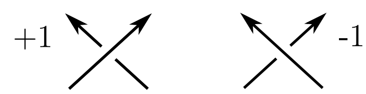


Figure 3. Right-hand and left-hand crossing of the knot

2.2 Knot Invariants

An important question to ask is whether or not two knots are the "same". We use knot invariants to distinguish between knots and classify them. We say that two knots are *equivalent* if they are related to each other by an ambient isotopy.

Definition 2.2.1. [13] A *knot invariant* is a property of a knot that does not change under ambient isotopy.

Definition 2.2.2. Links L_1 and L_2 in the three-sphere S^3 are equivalent if there is an orientation preserving piecewise linear homeomorphism $h : S^3 \to S^3$ such that $h(L_1) = L_2$.

An example of a knot invariant is the crossing number of a knot K, denoted $c(K)$. It is the least number of crossings that occurs in any projection of a knot [1]. The trefoil knot 1 has crossing number 3.

2.3 Pretzel Knots

For this thesis, we are interested in *pretzel knots*, specifically the family of pretzel knots $P(-2r, 2r+1, 2r+s)$ for $r \geq 1$ and $s \geq 3$. It is usually much more relevant to consider various classes of knots and links that have been found to be interesting, rather than to seek some list of all possible knots that would satisfy the Slope conjecture.

We consider the family $P(-2r, 2r+1, 2r+s)$, $r \geq 1$ and $s \geq 3$ more compelling, since the Slope conjecture has already been proven for alternating knots [5] and our family is non-alternating. An *alternating knot* is a knot with a diagram that has crossings that alternate between over and under as one travels around the knot in a fixed direction [1]. However, the family of pretzel knots $P(-2r, 2r+1, 2r+s)$ has no alternating diagrams.

Definition 2.3.1. Let $(a_1, a_2, ..., a_n)$ be a set of non-zero, positive or negative integers. A *pretzel link* $P(a_1, a_2, ..., a_n)$ is a link that has a diagram obtained by joining n twist regions side-by-side (see Figure 4). Each twist region is associated with an a_i and has either left- or right-hand crossings, as shown below. The a_i indicates the number of crossings in twist regions $1, 2, ..., n$ respectively. If a_i is positive then the twist region has right-hand crossings and if a_i is negative we will have left-hand crossings in the twist region.

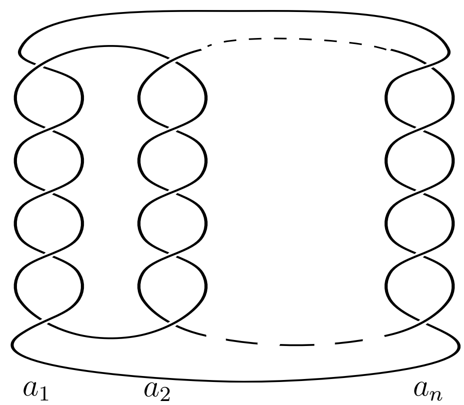


Figure 4. Pretzel knot $P(a_1, a_2, ..., a_n)$

In the figure above, the pretzel knot has all right-hand crosings. The family $P(-2r, 2r+1, 2r+3)$ for $r \in \mathbb{N}$ are considered pretzel knots, since they are a link of one component. This is because we have at most one twist region with an even crossing number (rest odd). If we have more than one twist region with even crossing number, then this gives a link. This is because the two twist regions with even crossing number combine in such a way that it "links" or "hooks" together (one crossing over, one crossing under) to form two disjoint, simple closed curves.

2.4 The Colored Jones Polynomial

The colored Jones polynomial is a powerful example of a knot invariant. In order to define the colored Jones polynomial we first define the Kauffman bracket polynomial, the Jones polynomial and the Chebychev polynomial.

Definition 2.4.1. [13] The *Kauffman bracket* is a function from unoriented link diagrams in the oriented plane to Laurent polynomials with integer coefficients in one variable A. It maps a diagram D to $\langle D \rangle \in \mathbb{Z}[A^{-1}, A]$ and is characterized by

1. $\langle \bigcirc \rangle = -A^2 - A^{-2}$

2. $\langle \times \rangle = A \langle)\,(\rangle + A^{-1} \langle \asymp \rangle$

Here $\langle D\rangle$ denotes the bracket polynomial of the knot K and A is a variable of the polynomial. In order to find the bracket polynomial of K, the first rule says that the polynomial of the trivial knot will be $-A^2 - A^{-2}$. The second rule replaces the crossing with a resolution.

Next, we define the writhe of a knot K, the Jones polynomial and then the Chebyshev polynomial.

Definition 2.4.2. [1] The *writhe* $w(D)$ of a knot diagram D is the sum of all the positive and negative crossings.

Definition 2.4.3. [13] The *Jones polynomial* $V_L(t)$ of an oriented link L is the Laurent polynomial in $t^{-1/2}$, with integer coefficients, defined by

$$V_L(t) = (-A)^{-w(D)}\langle D\rangle_{t^{1/2}=A^{-2}} \in \mathbb{Z}[t^{-1/2}, t^{1/2}]$$

where D is any oriented diagram for L and w is its writhe.

Note that we can talk about the Jones polynomial of a link since the polynomial is not dependent on the diagram D.

Definition 2.4.4. [10] For $n \geq 0$, the *Chebyshev polynomial* $S_n(x)$ is defined recursively as follows:

$$S_{n+2}(x) = xS_{n+1}(x) - S_n(x), \qquad S_1(x) = x, S_0(x) = 1$$

Now, we can define the colored Jones polynomial.

Definition 2.4.5. [10] For a knot $K \subset S^3$ let $J_{K,n}(t)$ denote the *n-th colored Jones polynomial of* K. For $n > 0$,

$$J_{K,n}(t) = ((-1)^{n-1}t^{(n^2-1)/4})^w(-1)^{n-1}\langle S_{n-1}(D)\rangle$$

Here $\langle S_{n-1}(D) \rangle$ is the linearly extended Kauffman bracket, D represents a link diagram and w is the writhe of the specific choice of diagram. D^n is interpreted as the n-blackboard cable of the link diagram D.

The colored Jones polynomial can detect more boundary slopes than the regular Jones polynomial - which computes boundary slopes only for alternating knots [2].

A simple example of $J_{K,3}(t)$ where K is the trefoil knot follows:

$$J_{K,3}(t) = (t^6)\langle S_2(D) \rangle$$

where $S_2(D) = D^2 - 1$. So then for this example,

$$\langle S_2(D) \rangle = \langle D^2 \rangle - 1$$

where D^2 is the following diagram:

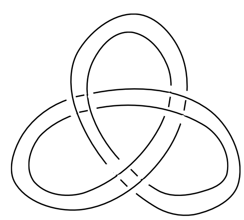


Figure 5. D^2-cable of the trefoil knot

2.5 The Slope Conjecture

Let K be a knot in S^3 and let $\mathring{N}(K)$ be a tubular neighborhood of K. Define the *knot complement* $M = S^3 - \mathring{N}(K)$.

Definition 2.5.1. [2] A compact, orientable, properly embedded surface $S \subset M$ is said to be *incompressible* if for any disc $D \subset M$ with $D \cap S = \partial D$, there exists a disc $D' \subset S$, with $\partial D' = \partial D$.

Definition 2.5.2. [2] A compact orientable, properly embedded surface $S \subset M$ is *∂-incompressible* if for each disc $D \subset M$ with $D \cap S = \partial_+ D$, and $D \cap \partial M = \partial_- D$. There is a disc $D' \subset S$ with $\partial_+ D' = \partial_+ D$ and $\partial_- D' \subset \partial S$.

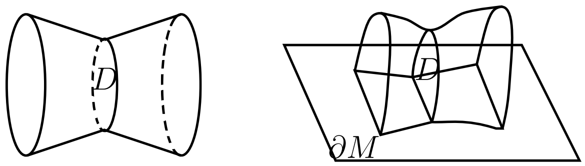


Figure 6. Left: A compressing disk. Right: A boundary compressing disk

Definition 2.5.3. [2] A compact, orientable, properly embedded surface $S \subset M$ is *essential* if it is both incompressible and ∂-incompressible. A compact non-orientable and properly embedded surface $S \subset M$ is essential if the induced inclusion map ι^* on fundamental groups:

$$\iota^* : \pi_1(S) \hookrightarrow \pi_1(M)$$

is injective.

Definition 2.5.4. [10] Let S be an essential surface with non-empty boundary in $\mathring{N}(K)$. A fraction $\frac{p}{q} \in \mathbb{Q} \cup \frac{1}{0}$ is a *boundary slope* of K if $p\mu + q\lambda$ represents the homology class of a component of ∂S in $H_1(\partial N(K))$. Here μ is the meridian and λ is the longitude.

The set of boundary slopes of a knot is an invariant of the knot, and we know the set is finite by [8].

Definition 2.5.5. [5] For a knot $K \subset S^3$ let $d_+[J_{K,n}(t)]$ and $d_-[J_{K,n}(t)]$ denote the *maximal* and *minimal degree of* $J_{K,n}(t)$ in t, respectively. The degrees $d_+[J_{K,n}(t)]$ and $d_-[J_{K,n}(t)]$ are quadratic quasi-polynomials. This means that, given a knot K, there is $n_K \in \mathbb{N}$ such that for all $n > n_K$ we have

$$d_+[J_{K,n}(t)] = a_K(n)n^2 + b_K(n)n + c_K(n)$$

$$d_-[J_{K,n}(t)] = a_K(n)n^2 + b_K(n)n + c_K(n)$$

where the coefficients are periodic functions from $\mathbb{N}$ to $\mathbb{Q}$ with finite integral period.

Definition 2.5.6. [5] For a knot K, define the *Jones slopes* js_K by:

$$js_K = \{\frac{2}{n^2}deg(J_{K,n}(t))|n \in \mathbb{N}\}$$

Let bs_K denote the set of boundary slopes of essential surfaces of K.

The Slope Conjecture asserts that the Jones slopes of any knot K are boundary slopes.

Conjecture 1. *[5]* (The Slope Conjecture) *For every knot we have*

$$2js_K \subset bs_K$$

This means that from the colored Jones polynomial, we can find a boundary slope for any knot using the degree of this polynomial.

CHAPTER III
THE DEGREES OF THE JONES POLYNOMIAL OF PRETZEL KNOTS

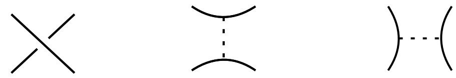
3.1 The Kauffman States of Pretzel Knots

In this section we prove our main theorem, which states that the minimum degree of the Jones polynomial of the family of pretzel knots $P(-2r, 2r+1, 2r+s)$ for $r \geq 1$ and $s \geq 3$ is $-6r - 3s + 3$. We begin by proving that the minimum degree of the Jones polynomial of $P(-2, 3, 7)$ as well as the $P(-4, 5, 7)$ pretzel knot is -18. We then use the techniques developed to generalize to a proof for the family of pretzel knots $P(-2r, 2r+1, 2r+s)$ for $r \geq 1$ and $s \geq 3$. Generally, we will label the three twist regions from left to right by a, b and c.

Definition 3.1.1. [13] A *state* s for a diagram D with n crossings labelled $1, ..., n$ is a function

$$s : \{1, 2, 3, ...n\} \rightarrow \{-1, 1\}$$

Given a link diagram D and a state s for D, sD is constructed from D by resolving each crossing into an A- or B-resolution according to whether $s(i)$ is equal to 1 or $s(i)$ is equal to -1 respectively. See the figure below.



$s(i) = 1$ $s(i) = -1$

Figure 7. The A and B resolutions of left-hand crossing

For the purpose of our research, we use the unreduced version of the Kauffman bracket, where the value of the bracket of the unknot is equal to $-A^{-2} - A^2$ instead of 1. We modify Lickorish's Proposition 5.1 accordingly:

If D is a link diagram with n crossings, the Kauffman bracket of D is given by

$$\langle D \rangle = \sum_s A^{\sum_{i=1}^n s(i)} (-A^{-2} - A^2)^{|sD|} \tag{III.1}$$

where the diagram sD, having no crossings is a set of disjoint simple closed curves and $\mid sD \mid$ is the number of circles after resolving the crossings.

Let s_+ and s_- be the two constant states such that s_+ has value $+1$ on every crossing and s_- has value -1 on every crossing. Fix the diagram $P(-2r, 2r+1, 2r+s)$ of a 3-tangle pretzel knot. We denote the three twist regions from left to right by a, b and c. Also, let $s_-\{k_a, k_b, k_c\}$ be the Kauffman state that chooses the A-resolution on k_a crossings in twist region a, k_b crossings in twist region b, and k_c crossings in twist region c, where k_a, k_b, k_c are positive integers $\leq$ the twist number. The Kauffman state chooses the B-resolution on all the rest of the crossings.

For example, $s_-\{1, 1, 1\}$, is the Kauffman state that chooses the A-resolution on one crossing in twist region a , b, and c, while the remaining crossings have B-resolution.

3.2 The $P(-2,3,7)$ Pretzel Knot

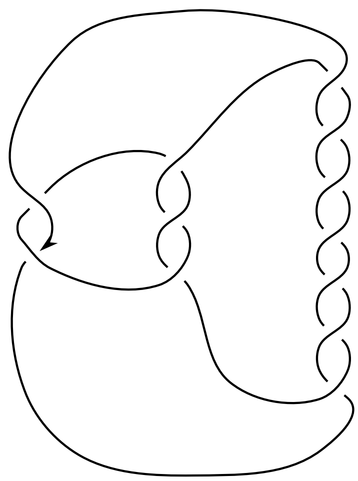


Figure 8. The P(-2,3,7) Pretzel knot

According to Lickorish, if two segments of s_+D that replace a crossing of D never belong to the same component of s_+D, then D is *plus-adequate*. Similarly, if two components of s_-D do not belong to the same component of s_-D then D is *minus-adequate* [13].

Lemma 1. *The diagram for the* $P(-2,3,7)$ *is plus-adequate, but not minus-adequate.*

Proof. We change D to s_+D by replacing all the crossings in the positive manner as shown in the figure below. In other words, s_+D is the Kauffman state that chooses the A-resolution on all crossings.

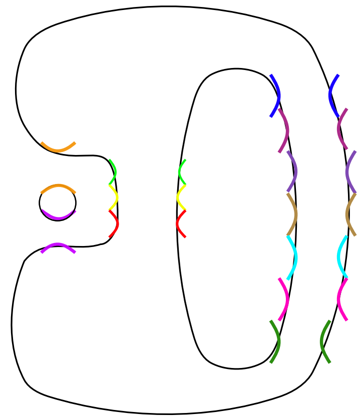


Figure 9. s_+D

According to Lickorish, if two segments of s_+D that replace a crossing of D never belong to the same component of s_+D, then D is *plus-adequate*. From the above figure 9, we see for each crossing, the two resulting arcs belong to different components. Thus, D is plus-adequate

Consider the Kauffman state s_-D, which chooses the B-resolution on all crossings.

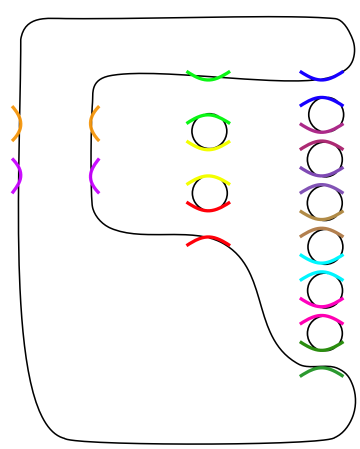


Figure 10. s_-D

From the above figure, we see that the purple and orange edge on twist region a have ends on the same circle. Therefore, there exists two segments of s_-D that belong to the same component of s_-D, and D is not minus-adequate. □

Let $M\langle D\rangle$ and $m\langle D\rangle$ denote the maximum and minimum powers of A that occur in the bracket (Laurent) polynomial of a diagram D.

We use the unreduced form of Lickorish's Lemma 5.4 to calculate the maximum $M\langle D\rangle$ degree of the Jones polynomial of the knot:

Lemma 2. *[13, Proposition 5.4] Let D be a link diagram with n crossings. Then $M\langle D\rangle \leq n + 2 \mid s_+D \mid$ with equality if D is plus-adequate, and $m\langle D\rangle \geq -n - 2 \mid s_-D \mid$ with equality if D is minus-adequate.*

Since our knot is plus-adequate, but not minus-adequate, we have that $M\langle D\rangle \leq n + 2 \mid s_+D \mid = 12 + 2 \mid 3 \mid = 18$. We set out to calculate the minimum degree of the knot, $m\langle D\rangle$.

In order to prove the main Theorem, we first prove the following Lemma:

Lemma 3. *Let x, h, and f be non-negative integers. Then*

$$\sum_{i=0}^{x} \binom{x}{i} A^{h+2i}(-A^2 - A^{-2})^{f+i} = (-1)^x A^{4x+h}(-A^2 - A^{-2})^f$$

Proof. Note,

$$\sum_{i=0}^{x} \binom{x}{i} A^{h+2i}(-A^2 - A^{-2})^{f+i} = A^h(-A^2 - A^{-2})^f \sum_{i=0}^{x} \binom{x}{i} A^{2i}(-A^2 - A^{-2})^i$$
$$= A^h(-A^2 - A^{-2})^f \sum_{i=0}^{x} \binom{x}{i} (-A^4 - 1)^i$$

Using the Binomial Theorem, we have

$$= A^h(-A^2 - A^{-2})^f[(-A^4 - 1) + 1]^x = (-1)^x A^{4x+h}(-A^2 - A^{-2})^f$$

□

Recall by (III.1) that $\langle D \rangle$ is defined by summing over polynomials $A^{\sum_{i=1}^{n} s(i)}(-A^{-2} - A^{2})^{|sD|}$ associated to each Kauffman state. From this definition it is clear that we are interested in $\sum s(i)$ and the number of circles $|s_D|$.

From the proof of Lemma 5.4 in Lickorish, we know that any state s can be achieved by starting with s_+ and changing, one at a time, the value of s_+ on selected integers that label the crossings [13]. Similarly, any state can be achieved by starting with s_- and resolving each crossing one at a time. This also means that for states $s_0, s_1, ..., s_k$, $s_{q-1}D$ and s_qD for $q = 1, 2, ...k$ are the same diagram except near one crossing of D, and $|s_qD| = |s_{q-1}D| \pm 1$.

Therefore, since the minimum degree of the polynomial associated to the Kauffman state s is equal to $\sum s(i) - 2|sD|$ and $\sum s_q(i) \geq \sum s_-(i) + 2$, we have that the degree of s_- will always be smaller than or equal to the degrees of all other states.

Theorem 1. *The minimum degree of the Jones polynomial for the pretzel knot* $P(-2, 3, 7)$ *is* $m\langle D \rangle = -18$ *with coefficient* 1.

Proof. We start with s_- and find all the other states that have the same degree as it by changing the resolutions on crossings from B to A. The Kauffman state s_- has degree -30. This is because the number of crossings is 12, the sum of $s_-(i) = -12$ and the number of circles in s_- is 9, which gives that $-12 - 2(9) = -30$. We consider cases between -30 and -18 and show by direct calculation, with the help of lemma 3, that the minimum degree of the Jones polynomial is -18 with coefficient 1.

We organize these states by the number of crossings chosen in twist region b and c and then group the number of crossings in a with the same degree.

0 crossings chosen in twist region b and c:

Note that, if zero crossings are chosen in twist regions b and c, no matter which crossing we choose to change to an A-resolution on twist region a, the state we end up getting will have the same degree as s_-. This is because no matter which crossing we choose to change, the circles $|sD|$ on twist region a will merge. More precisely, let U be a subset of the crossings of twist region a, and let s_U be the Kauffman state obtained from s_- by changing the resolution on crossings in U from B to A. Then $|s_U D| = |s_- D| + |U|$.

In this situation the sum of all the polynomials associated to states that choose 0 crossings in twist region b and c with degree $= -30$ is the same as $\sum_{i=0}^{2} A^{-12+2i}(-A^2 - A^{-2})^{9+i}$.

We apply Lemma 3, with $x = 2$, $f = 9$, $m = 0$ and $h = -12$.

Then,

$$\begin{aligned}\sum_{i=0}^{2} A^{-12+2i}(-A^2 - A^{-2})^{9+i} &= (-1)^2 A^{4(2)+(-12)}(-A^2 - A^{-2})^9 \\ &= A^{-4}(-A^2 - A^{-2})^9\end{aligned}$$

1 or 2 crossings chosen in twist region b and c: Similar to the case where 0 crossings are chosen in twist region b and c, if a state has 1 or 2 crossings chosen in twist regions b and c its degree is -26 and -22 respectively, regardless of which subsets we choose to change the resolution from B to A in twist region a. We account for the number of states by multiplying the polynomial by $\binom{10}{1}$ and $\binom{10}{2}$ respectively since there are 10 crossings total in b and c.

The polynomials resulting from these two cases are shown below.

(1 crossing case) $$\begin{aligned}&\binom{10}{1}[\sum_{i=0}^{2} A^{-10+2i}(-A^2 - A^{-2})^{8+i}] \\ &= \binom{10}{1}[(-1)^2 A^{4(2)+(-10)}(-A^2 - A^{-2})^8] = \binom{10}{1}[A^{-2}(-A^2 - A^{-2})^8]\end{aligned}$$

$$\text{(2 crossing case)} \quad \binom{10}{2}[\sum_{i=0}^{2} A^{-8+2i}(-A^2 - A^{-2})^{7+i}]$$
$$= \binom{10}{2}[(-1)^2 A^{4(2)+(-8)}(-A^2 - A^{-2})^7] = \binom{10}{2}(-A^2 - A^{-2})^7]$$

3 crossings chosen in twist regions b and c. We account for the number of states by multiplying the polynomial by $\binom{10}{3}$. Except for the state where all 3 crossings are chosen to have A-resolutions on twist region b, we have that resolving crossings in twist region a will split the same number of circles as the number of crossings in the subset.

$$\text{(3 crossing case 1)} \quad [\binom{10}{3} - 1][\sum_{i=0}^{2} A^{-6+2i}(-A^2 - A^{-2})^{6+i}]$$
$$= [\binom{10}{3} - 1][(-1)^2 A^{4(2)+(-6)}(-A^2 - A^{-2})^6]$$
$$= [\binom{10}{3} - 1][A^2(-A^2 - A^{-2})^6]$$

The case of the Kauffman state that chooses all crossings in twist region a to have A-resolution and B-resolution for the other crossings is shown below. We multiply the polynomial by $\binom{3}{3}$ and $\binom{7}{0}$ to account for the single case. This is because we choose all crossings in twist region b to have A-resolution and none of the other crossings in twist region c. The degree of one of the terms is -22.

$$\text{(3 crossing case 2)} \quad \binom{3}{3}\binom{7}{0}[A^{-6}(-A^2 - A^{-2})^8]$$

The next states that have degrees ≤ -18 are listed below. The first state to consider is the Kauffman state that chooses the A-resolution for all the crossings in twist region b, as well as one crossing in twist region c, all other crossings have B-resolution. We multiply the polynomial by $\binom{3}{3}$ and $\binom{7}{1}$ to account for these seven cases.

$$s_{-}\{0,3,1\} \qquad \binom{3}{3}\binom{7}{1}[A^{-4}(-A^{2}-A^{-2})^{7}]$$

Next, consider the Kauffman state that chooses the A-resolution for all the crossings in twist region b, as well as one crossing in twist region a, all other crossings have B-resolution. We multiply the polynomial by $\binom{2}{1}$ and $\binom{3}{3}$ to account for these two cases.

$$s_{-}\{1,3,0\} \qquad \binom{2}{1}\binom{3}{3}[A^{-4}(-A^{2}-A^{-2})^{7}]$$

Finally, consider the Kauffman state that chooses the A-resolution for all the crossings in twist regions a and b and B-resolution for all crossings in twist region c. We multiply the polynomial by $\binom{2}{1}$ and $\binom{3}{3}$ to account for these two cases.

$$s_{-}\{2,3,0\} \qquad \binom{2}{2}\binom{3}{3}[A^{-2}(-A^{2}-A^{-2})^{8}]$$

Let d denote the degree of the Kauffman state s and d_s the degree of the sum over all states that satisfy the conditions $0 \leq k_a \leq 2$ and $k_b + k_c$ equal to the number of crossing(s) chosen to have A-resolution in b and c for the state $s_{-}\{k_a, k_b, k_c\}$. To summarize,

Table 1. Degrees of States of $P(-2,3,7)$

d	d_s	State	Polynomial
-30	-22	$0 \leq k_a \leq 2$ and $k_b + k_c = 0$	$\binom{10}{0} A^{-4}(-A^2 - A^{-2})^9$
-26	-18	$0 \leq k_a \leq 2$ and $k_b + k_c = 1$	$\binom{10}{1} A^{-2}(-A^2 - A^{-2})^8$
-22	-22	$s_-\{0,3,0\}$	$\binom{3}{3}\binom{7}{0} A^{-6}(-A^2 - A^{-2})^8$
-18	-18	$s_-\{1,3,0\}$	$\binom{2}{1}\binom{3}{3}\binom{7}{0} A^{-4}(-A^2 - A^{-2})^7$
-18	-18	$s_-\{2,3,0\}$	$\binom{2}{2}\binom{3}{3}\binom{7}{0} A^{-2}(-A^2 - A^{-2})^8$
-18	-18	$s_-\{0,3,1\}$	$\binom{3}{3}\binom{7}{1} A^{-4}(-A^2 - A^{-2})^7$

Note that the 2 crossing case is not included in the table, since the minimum degree of the polynomial is larger than -18.

We have accounted for all the cases with three crossings in twist regions b and c. The only case with four crossings in twist region b and c that we need to consider is $s_-\{0,3,1\}$, because other cases have degree bigger than -18.

Finally, we sum the polynomials in the table:

Table 2. Sum of the coefficients of the A^{-22} and A^{-18} terms for $P(-2,3,7)$

Polynomial	Coeffiecient of A^{-22} term	Coeffiecient of A^{-18} term
$\binom{10}{0} A^{-4}(-A^2 - A^{-2})^9$	-1	-9
$\binom{10}{1} A^{-2}(-A^2 - A^{-2})^8$		10
$\binom{3}{3}\binom{7}{0} A^{-6}(-A^2 - A^{-2})^8$	1	8
$\binom{2}{1}\binom{3}{3}\binom{7}{0} A^{-4}(-A^2 - A^{-2})^7$		-2
$\binom{2}{2}\binom{3}{3}\binom{7}{0} A^{-2}(-A^2 - A^{-2})^8$		1
$\binom{3}{3}\binom{7}{1} A^{-4}(-A^2 - A^{-2})^7$		-7
	0	1

Since all of the states with degree less than or equal to -18 have either been included or exhausted, we have that the minimum degree of the pretzel knot $P(-2,3,7)$ is $m\langle D\rangle = -18$ with coefficient 1.

□

3.3 The P(-4,5,7) Pretzel Knot

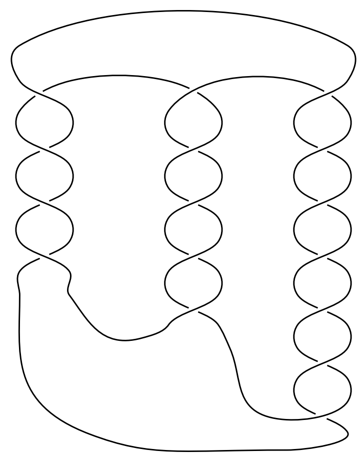


Figure 11. The P(-4,5,7) Pretzel knot

Theorem 2. *The minimum degree of the Jones polynomial for the pretzel knot $P(-4,5,7)$ is $m\langle D\rangle = -18$ with coefficient 1.*

Proof. Similarly to proof of Theorem 1, we begin with s_- and find all the other states that have the same degree as it by changing the resolutions on crossings from A to B. The Kauffman state s_- has degree -38 for this knot. This is because the number of crossings is 16, the sum of $s_-(i) = -16$ and the number of circles in s_- is 11 which gives that $-16 - 2(11) = -38$. We consider cases between -38 and -18 and show by direct calculation, with the help of lemma 3, that the minimum degree of the Jones polynomial is -18 with coefficient 1.

We organize the states, the same as before, by the number of crossings chosen in twist region b and c and then group the number of crossings in a with the same degree. Again, let d denote the degree of the Kauffman state s and d_s the degree of the sum over all states that satisfy the conditions $0 \leq k_a \leq 4$ and $k_b + k_c$ equal to the number of crossing(s) chosen to have A-resolution in b and c for the state $s_-\{k_a, k_b, k_c\}$.

We consider cases for $k_b + k_c \leq 5$ and we do not consider the cases for $1 < k_b + k_c \leq 4$ because the degree will be larger than -18. We summarize the results,

Table 3. Degrees of States for $P(-4, 5, 7)$

d	d_s	State	Polynomial
-38	-22	$0 \leq k_a \leq 4$ and $k_b + k_c = 0$	$\binom{12}{0}(-A^2 - A^{-2})^{11}$
-34	-18	$0 \leq k_a \leq 4$ and $k_b + k_c = 1$	$\binom{12}{1}A^2(-A^2 - A^{-2})^{10}$
-22	-22	$s_-\{0, 5, 0\}$	$\binom{4}{0}\binom{5}{5}\binom{7}{0}A^{-6}(-A^2 - A^{-2})^8$
-18	-18	$s_-\{0, 5, 1\}$	$\binom{4}{0}\binom{5}{5}\binom{7}{1}A^{-4}(-A^2 - A^{-2})^7$
-18	-18	$s_-\{1, 5, 0\}$	$\binom{4}{1}\binom{5}{5}\binom{7}{0}A^{-4}(-A^2 - A^{-2})^7$
-18	-18	$s_-\{2, 5, 0\}$	$\binom{4}{2}\binom{5}{5}\binom{7}{0}A^{-2}(-A^2 - A^{-2})^8$
-18	-18	$s_-\{3, 5, 0\}$	$\binom{4}{3}\binom{5}{5}\binom{7}{0}(-A^2 - A^{-2})^9$
-18	-18	$s_-\{4, 5, 0\}$	$\binom{4}{4}\binom{5}{5}\binom{7}{0}A^2(-A^2 - A^{-2})^{10}$

We sum the polynomials:

Table 4. Sum of the coefficients of the A^{-22} and A^{-18} terms for $P(-4,5,7)$

Polynomial	Coeffiecient of A^{-22} term	Coeffiecient of A^{-18} term
$\binom{12}{0}(-A^2-A^{-2})^{11}$	-1	-11
$\binom{12}{1}A^2(-A^2-A^{-2})^{10}$		12
$\binom{4}{0}\binom{5}{5}\binom{7}{0}A^{-6}(-A^2-A^{-2})^8$	1	8
$\binom{4}{0}\binom{5}{5}\binom{7}{1}A^{-4}(-A^2-A^{-2})^7$		-7
$\binom{4}{1}\binom{5}{5}\binom{7}{0}A^{-4}(-A^2-A^{-2})^7$		-4
$\binom{4}{2}\binom{5}{5}\binom{7}{0}A^{-2}(-A^2-A^{-2})^8$		6
$\binom{4}{3}\binom{5}{5}\binom{7}{0}(-A^2-A^{-2})^9$		-4
$\binom{4}{4}\binom{5}{5}\binom{7}{0}A^2(-A^2-A^{-2})^{10}$		1
	0	1

Since all of the states with degree less than or equal to -18 have either been included or exhausted, we have that the minimum degree of the pretzel knot $P(-4,5,7)$ is $m\langle D\rangle = -18$ with coefficient 1.

□

3.4 The Family of Pretzel Knots $P(-2r, 2r+1, 2r+s)$

Now we set out to prove the general case for the pretzel knot. Let $x = 2r, y = 2r+1, z = 2r+s$ for $r \leq 1, s \leq 3$ be the number of crossings in twist region a, b, and c respectively. The Kauffman state s_- for the general case has polynomial with degree $-x-3y-3z-2$. This is because the number of crossings is $(x+y+z)$, the sum of $s_-(i) = -(x+y+z)$ and the number of circles in s_- is $y+z-1$ which gives that $-(x+y+z)-2(y+z-1) = -x-3y-3z-2$. We consider cases between $-x-3y-3z-2$ and $-6r-3s+3$ and show that the minimum degree of the Jones polynomial of the family of pretzel knots $P(-2r, 2r+1, 2r+s)$ is -18 with coefficient 1.

Theorem 3. *The minimum degree of the pretzel knot $P(-2r, 2r+1, 2r+s)$ for $r \geq 1$ and $s \geq 3$ is $m\langle D\rangle = -6r - 3s + 3$ with coefficient 1 .*

Proof. Consider the table for the pretzel knot $P(-2r, 2r+1, 2r+s)$

Table 5. Degrees of States for the family of pretzel knots $P(-2r, 2r+1, 2r+s)$

d	d_s	State	Polynomial
$-14r-3s-1$	$2k_a - 14r - 3s - 1$	$0 \leq k_a \leq x$ and $k_b + k_c = 0$	$\binom{b+c}{0}A^{2k_a-x-y-z}(-A^2 - A^{-2})^{(y+z)-1}$
$-14r-3s+3$	$2k_a - 14r - 3s + 3$	$0 \leq k_a \leq x$ and $k_b + k_c = 1$	$\binom{b+c}{1}A^{2k_a-x-y-z+2}(-A^2 - A^{-2})^{(y+z)-2}$
$-2r-s-2$	$-2r-s-2$	$s_-\{0, y, 0\}$	$\binom{y}{y}\binom{z}{0}A^{x-y+z}(-A^2 - A^{-2})^{z+1}$
$-6r-3s+3$	$-6r-3s+3$	$s_-\{k, y, 0\}$ and $1 \leq k \leq x$	$\binom{x}{k}\binom{y}{y}\binom{z}{0}A^{y+2k-x-z}(-A^2 - A^{-2})^{z+k-1}$
$-6r-3s+3$	$-6r-3s+3$	$s_-\{0, y, 1\}$	$\binom{y}{y}\binom{z}{1}A^{-x+y-z+2}(-A^2 - A^{-2})^{z}$

Table 6. Sum of the coefficients of the A^{-22} and A^{-18} terms
for $P(-2r, 2r+1, 2r+s)$

State	Coeffiecient of A^{-22} term	Coeffiecient of A^{-18} term
$0 \le k_a \le x$ and $k_b + k_c = 0$	-1	$-(y+z-1)$
$0 \le k_a \le x$ and $k_b + k_c = 1$	0	$y+z$
$s_-\{0, y, 0\}$	1	$z+1$
$s_-\{0, y, 1\}$	0	$-z$
$s_-\{k, y, 0\}$ for $1 \le k \le x$	0	-1
	0	1

The coefficients of the A^{-22} term add up to 0, while the coefficients of the A^{-18} term add up to 1. We do not consider other states since then the degree of the polynomial will be more than -18. □

CHAPTER IV
NORMAL SURFACE THEORY

The difficulty in resolving the slope conjecture lies in determining the Jones slope and in determining boundary slopes of a knot. From the definition of the colored Jones polynomial, it is clear that it is very difficult to compute due to the exponential complexity on the crossings of the knot diagram. On the other hand, computing the boundary slopes of a knot has been solved with algorithms involving normal surface theory on triangulated 3-manifolds.

We define a topological manifold M as follows:

Definition 4.0.1. [9] A *Topological Manifold M* is a topological space with a family of open sets and functions $\{(M_\alpha, f_\alpha)\}$ such that,

$$M = \cup_\alpha M_\alpha$$

and

$$\forall \alpha, \phi_\alpha : M_\alpha \to U_\alpha$$

is a homeomorphism onto an open subset $U_\alpha \subset \mathbb{R}^n$

Some examples of manifolds are the circle S^1, the two-sphere S^2 and the torus T^2 shown in the figure below.

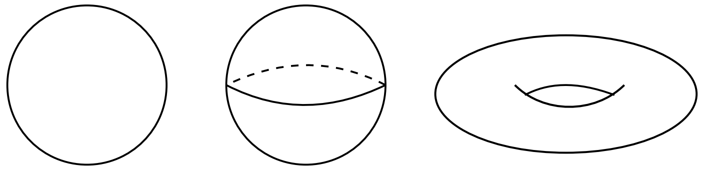


Figure 12. Circle S^1, sphere S^2, and Torus T^2

The *standard n-simplex* is given as

$$\triangle^n = \{(t_0, ..., t_n) \in \mathbb{R}^{n+1} | \sum_i t_i = 1, t_i \geq 0, \forall i\}$$

whose vertices are the unit vectors along the coordinate axes. This means that a 0-simplex is a point, a 1-simplex is a line segment in two-dimensional space, a 2-simplex is a triangle in three-dimensional space. We are mostly considered with the 3-simplex tetrahedron.

Definition 4.0.2. [9] A triangulated n-manifold is a pair (M, C), where M is a topological manifold of dimension n and C a simplicial complex such that

- $|C| = M$
- For every compact subset $A \subset M$, the set $\{ f \in C | Im(f) \cap A \neq \emptyset \}$ is finite.

We call C a triangulation of M. The union of the images of the simplices in C is denoted by $|C|$.

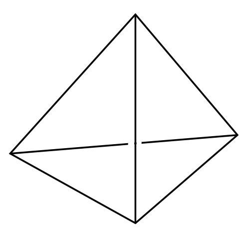


Figure 13. Triangulation of the 3-ball

It is known that all compact 3-manifolds, which includes complements of knots, admit a triangulation [15]. A *normal surface* F is a surface embedded in a triangulated compact 3-manifold that intersects each tetrahedron in normal disks which are either triangles or quadrilaterals [7].

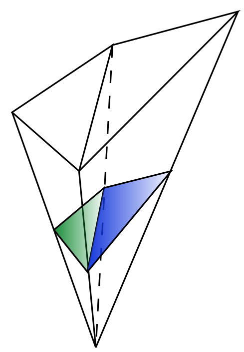


Figure 14. Triangulated manifold with normal surface

The set of isotopy classes of normal surfaces of a compact, triangulated 3-manifold is finite, and every essential surface is isotopic to a Haken sum of normal surfaces [14]. Moreover, by counting the intersection of the boundary of normal disks with tetrahedra in the triangulation, normal surfaces can be described as coordinate vectors satisfying a linear system of equations. This reduces the problem of finding boundary slopes to the linear algebra of finding normal surfaces given a triangulation on the knot complement.

A *normal isotopy* is an isotopy which is invariant on each simplex of C [17]. The normal isotopy class of an elementary disk in a tetrahedron is called a *disk type* and the normal isotopy class of a spanning arc in a 2-simplex of a tetrahedron is called an *arc type* [17]. In each tetrahedron there are seven disk types: four triangles or T-disks and three quadrilaterals or Q-disks.

There are 3 types of Q-discs:

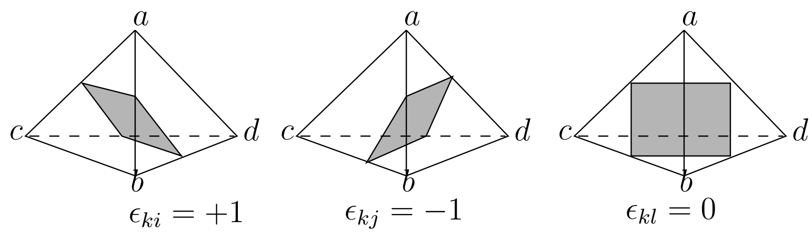


Figure 15. Types of Q-disks

The Q-disks are labeled according to the slope of the terahedra with respect to the edge [a,b]. Let t denote the number of tetrahedra in a triangulation of the manifold. If we fix an ordering $d_1^Q, ..., d_{3t}^Q$ and $d_1^T, ..., d_{4t}^T$ of the disc types in C, where the Q-disc types d_i^Q are listed first followed by the T-disk types d_j^T. A $7t$-tuple $\overrightarrow{F} = (x_1, ..., x_{3t}, y_1, ..., y_{4t})$ called the *normal coordinates of* F, is assigned to a normal surface F by letting x_i denote the number of elementary Q-disks in F of type d_i^Q and y_j denote the number of elementary T-disks in F of type d_j^T.

Tollefson, in Normal Suraface Q-theory, describes an approach to normal surface theory for triangulated 3-manifolds which uses only the quadrilateral disc types (Q-discs) to represent a nontrivial normal surface. Tollefson shows that it is possible to represent normal surfaces using only coordinates of quadrilateral disks.

Theorem 4. *[17] Let M be a compact 3-manifold with a fixed triangulation K. If F is a normal surface in M then the Q-coordinates $\overrightarrow{\mathrm{F_Q}}$ give an admissible solution to the Q-matching equations. Moreover, if $\overrightarrow{\mathrm{z}}$ is a nonzero admissible solution to the Q-matching equations then there exists a unique normal surface F in M with no trivial components such that $\overrightarrow{\mathrm{F_Q}} = \overrightarrow{\mathrm{z}}$.*

If F is a normal surface then $\overrightarrow{F}_Q = (x_1, ...x_{3t})$ satisfies the following linear system of equations, one equation for each interior 1-simplex e_k of C.

$$\sum_{i=1}^{3t} e_{k,i} x_i = 0$$

for $0 \leq x_i$, $1 \leq i \leq 3t$. These equations are called *Q-matching equations* and must equal 0 so that we have no dangling edges (the corner of each disks must perfectly aligned/matched).

A connected normal surface containing only elementary T-disks is a 2-sphere or disk frontier of a regular neighborhood of a vertex of C and will be referred to as a *trivial surface* [17]. A non-negative integral solution $(x_1, ..., x_{3t})$ of the Q-matching equations is *admissible* if it has the property that for each tetrahedron of C at most one of the three variables x_i corresponding to Q-disk types in the tetrahedron is nonzero.

We consider Lens Spaces that form a class of examples of 3-manifolds $L_{p,q}$ parameterized by two coprime integers p and q. Consider a regular planar p sided polygon P, together with two points n and s above and below. We connect each vertex of P to n and s forming a bipyramid that we then fill in. Label each edge of P, $e_0, ...e_{p-1}$ and label the corresponding triangular faces above and below as n_i and s_i. Form a quotient space by identifying n to s and the triangular faces n_i with $s_{i+q} mod\ p$. This forms a closed 3-manifold [9].

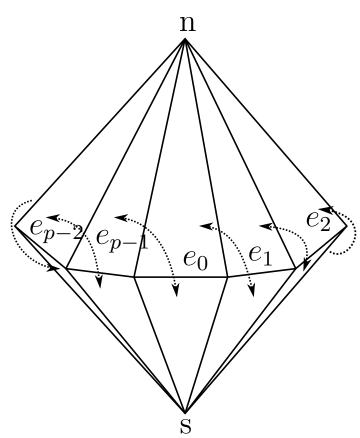


Figure 16. The Lens Space $L_{p,1}$

Consider the space $L_{3,1}$ (which is two tetrahedra glued):

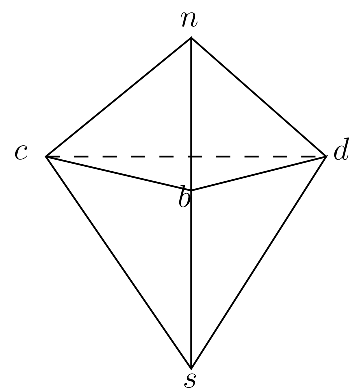


Figure 17. The Lens Space $L_{3,1}$

In $L_{3,1}$ the Q-disks are labeled according to the slope of the top tetrahedra with respect to the edge $[n, b]$ and the bottom tetrahedra with respect to $[b, s]$. When we identify the triangular faces to form the lens space, we will match the following edges to form 3 equivalence classes:

1. $[n, b]$, $[s, d]$, $[s, c]$
2. $[b, c]$, $[c, d]$, $[b, d]$
3. $[n, c]$, $[b, s]$, $[n, d]$

We write the Q-matching equations for $L_{3,1}$: Since we have 3 types of Q-discs as well as the case of no Q-discs in both top and bottom tetrahedron, we have $4 * 4 = 16$ total cases. We consider only 9 of these cases, since the other seven cases have either one or both tertrahedra with no Q-disc. These seven cases will not satisfy the Q-matching equations since they must equal 0. The remaining 9 cases are summarized in the table below. Let $\triangle$ denote tetrahedra.

Table 7. Q-matching equations for $L_{3,1}$

Type of Q-disc in top $\triangle$	Type of Q-disc in bottom $\triangle$	Satisfies/Does not satisfy the Q-matching equations
$+1$	$+1$	satisfies
$+1$	-1	satisfies
$+1$	0	does not satisfy
-1	$+1$	satisfies
-1	-1	satisfies
-1	0	does not satisfy
0	$+1$	does not satisfy
0	-1	does not satisfy
0	0	does not satisfy

CHAPTER V

CONCLUSION

An immediate impact of this research is the better understanding of the behavior of the family of pretzel knots $P(-2r, 2r+1, 2r+s)$ for $r \geq 1$ and $s \geq 3$. We have proven consistent cancellation in order to determine the minimum degree of the Jones polynomial for this family.

Since the colored Jones polynomial of pretzel knots are building blocks of the colored Jones polynomial for general classes of knots, the insights gained have the potential of generalizing the colored Jones polynomial. In other words, the evaluation of the Jones polynomial is expected to generalize to the computation of the colored Jones polynomial. This could be used to prove that the family of pretzel knots $P(-2r, 2r+1, 2r+s)$ for $r \geq 1$ and $s \geq 3$ satisfies or disproves the Slope conjecture.

The goal is to directly relate the normal surface theory to the number-theoretic evaluation of the colored Jones polynomial. The hope is to find parallels between this process and the process of evaluating the colored Jones polynomial, thereby proving a direct connection between the polynomial and essential surfaces in the complement of a knot.

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BIOGRAPHICAL SKETCH

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