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Chief Factors and the Principal Block of a Restricted Lie Algebra

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Let L be a finite-dimensional restricted Lie algebra over a commutative field $\mathbb{F}$ of characteristic p, and let $u(L)$ denote the restricted universal enveloping algebra of L (cf. **[SF**, Theorem 2.5.1**]**). A complete classification of simple restricted L-modules (over an algebraically closed field) exists for nilpotent restricted Lie algebras and restricted simple Lie algebras of classical and Cartan type, and their distribution into blocks is also known (see **[Voigt]**, **[Fe4]**, **[Hu]**, and **[HN]**). A *block decomposition* of $u(L)$ is a decomposition

$$u(L) = \bigoplus_{j=0}^{r} B_j(L),$$

into indecomposable two-sided ideals $B_j(L)$ of $u(L)$, the so-called *block ideals* of $u(L)$, which are finite-dimensional associative $\mathbb{F}$-algebras with an identity element (see **[HB**, Theorem VII.12.1**]**). Every indecomposable $u(L)$-module is a (unitary, left) $B_j(L)$-module for some uniquely determined j, and in particular, this induces an equivalence relation "*belonging to the same block*" on the finite set $\mathrm{Irr}_p(L)$ of all isomorphism classes of (irreducible or) simple restricted L-modules

$$\mathrm{Irr}_p(L) = \bigcup_{j=0}^{r} \mathbb{B}_j(L),$$

such that each equivalence class $\mathbb{B}_j(L) = \{[S] \in \mathrm{Irr}_p(L) \mid B_j(L) \cdot S = S\}$, a so-called *block* of $u(L)$, is in one-to-one correspondence with the set of isomorphism classes of simple $B_j(L)$-modules. For more information on block theory we refer the reader to **[CR**, §55**]** and **[HB**, Chapter VII**]** or **[Fe4]** and **[Fe5]**.

The block of $u(L)$ containing the one-dimensional trivial L-module is called the *principal block* of L, and the labeling will be chosen such that $\mathbb{B}_0(L)$ is the principal block. The principal block turns out to be the most complicated block of L (see for example **[Fe4**, Corollary 1, Examples 1 and 2**]**).

The purpose of this note is to apply the methods of **[FK]** in order to relate the projective indecomposable modules and the simple modules of the principal block ideal of a finite-dimensional restricted Lie algebra to tensor products (over the ground field) of (finitely many) composition factors of the adjoint module. The latter can be read off from the structure constants of the Lie algebra and therefore are easily accessible in given examples. Moreover, at the same time we also obtain a different approach to the results of **[Voigt]** and **[Fe4]** for supersolvable (respectively nilpotent) restricted Lie algebras and a Lie theoretic analogue of a theorem of Alperin **[Alp**, Theorem 1**]** for p-semiprimitive restricted Lie algebras.

It is common from group theory to call a composition factor of the adjoint module of a finite-dimensional Lie algebra L a *chief factor* of L (see **[Ba1]**), and we denote the set of isomorphism classes of chief factors of L by $\mathcal{F}(L)$. If $\mathrm{Ann}_L(M) := \{x \in L \mid x \cdot M = 0\}$ is the *annihilator* of an arbitrary L-module M, then the largest nilpotent ideal $\mathrm{Nil}(L)$ of a finite-dimensional Lie algebra L can be described as follows.

Proposition 1. *Let L be a finite-dimensional Lie algebra. Then*

$$\mathrm{Nil}(L) = \bigcap_{S \in \mathcal{F}(L)} \mathrm{Ann}_L(S).$$

Proof. If $y \in \mathrm{Nil}(L)$, then y is ad-nilpotent with respect to $\mathrm{Nil}(L)$, i.e., $\mathrm{ad}_{\mathrm{Nil}(L)}\, y$ is nilpotent. Hence for any $y \in \mathrm{Nil}(L)$ there exists a positive integer n such that $(\mathrm{ad}_{\mathrm{Nil}(L)}\, y)^n = 0$, and therefore we obtain for every $x \in L$:

$$(\mathrm{ad}_L\ y)^{n+1}(x) = (\mathrm{ad}_L\ y)^n([y, x]) \in (\mathrm{ad}_L\ y)^n(\mathrm{Nil}(L)) = 0.$$

This shows that y is in fact ad-nilpotent on L. In particular, $\mathrm{Nil}(L)$ acts nilpotently on every chief factor S of L. Then the Engel-Jacobson theorem implies

$$S^{\mathrm{Nil}(L)} \neq 0,$$

and the simplicity of S yields $\mathrm{Nil}(L) \subseteq \mathrm{Ann}_L(S)$.

In order to show the other inclusion, let

$$L = L_0 \supset L_1 \supset \cdots \supset L_n = 0$$

be a composition series of L. If $x \in L$ acts trivially on every chief factor of L, then $[x, L_j] \subseteq L_{j+1}$ for every $0 \leq j \leq n-1$. As a result, $(\mathrm{ad}_L\ x)^n(L) \subseteq L_n = 0$, i.e., by virtue of Engel's theorem, $\bigcap_{S \in \mathcal{F}(L)} \mathrm{Ann}_L(S)$ is a nilpotent ideal of L and thus contained in $\mathrm{Nil}(L)$. □

Proposition 1 enables us to prove the first main result of this note.

Theorem 1. *Let L be a finite-dimensional restricted Lie algebra. Then every projective indecomposable $u(L/\mathrm{Nil}(L))$-module is a direct summand of a suitable tensor product of (finitely many) chief factors of L.*

Proof. Set

$$V := \bigoplus_{S \in \mathcal{F}(L)} S.$$

According to Proposition 1, we have

$$\mathrm{Ann}_L(V) = \bigcap_{S \in \mathcal{F}(L)} \mathrm{Ann}_L(S) = \mathrm{Nil}(L).$$

Hence V is a faithful $L/\operatorname{Nil}(L)$-module. Then **[FK**, Corollary 3.2**]** in conjunction with **[FK**, Theorem 2.4**]** implies that $\mathcal{T}_n(V)$ is a faithful $u(L/\operatorname{Nil}(L))$-module for some positive integer n ($\leq \dim_{\mathbb{F}} u(L/\operatorname{Nil}(L))$). Since $\mathcal{T}_n(V)$ is a direct sum of certain tensor products of (at most n) chief factors of L, the assertion follows from **[FK**, Remark 1.4**]**. □

Remark. Note that the projective indecomposable $u(L)$-modules are *not* necessarily direct summands of a tensor product of chief factors of L. For example, in view of the main result of **[Ho]**, the projective cover $P_L(\mathbb{F})$ of the one-dimensional trivial module $\mathbb{F}$ of a finite-dimensional nilpotent restricted Lie algebra L is a direct summand of such a tensor product (i.e., in this case $\mathbb{F}$ itself) if and only if L is a torus.

In particular, we obtain for $\operatorname{Nil}(L) = 0$:

Corollary 1. *Let L be a finite-dimensional semisimple restricted Lie algebra. Then every projective indecomposable $u(L)$-module is a direct summand of a suitable tensor product of (finitely many) chief factors of L.* □

Remark. Note that for a restricted *simple* Lie algebra Corollary 1 is an immediate consequence of **[FK**, Theorem 3.3(a)**]** because the only chief factor is the adjoint module which itself is faithful.

Using

$$\operatorname{Rad}_p(L) = \bigcap_{S \in \operatorname{Irr}_p(L)} \operatorname{Ann}_L(S)$$

(see **[SF**, Theorem 5.5.3(2)**]**), it is possible to replace in Theorem 1 the largest nilpotent ideal $\operatorname{Nil}(L)$ by the largest p-nilpotent ideal $\operatorname{Rad}_p(L)$ if we allow in the conclusion that *any* simple restricted L-module can occur in the tensor product. Recall that a restricted Lie algebra is called *p-semiprimitive* if $\operatorname{Rad}_p(L) = 0$ (see **[SF**, p. 220**]**). Then the just mentioned analogue of Theorem 1 immediately implies that for a finite-dimensional p-semiprimitive restricted Lie algebra L every projective indecomposable $u(L)$-module is a direct summand of a suitable tensor product of (finitely many) simple restricted L-modules (see **[Alp**, Theorem 1**]** for the analogue in the modular representation theory of finite groups).

In order to derive our second main result from Theorem 1, we need the following necessary condition for a simple restricted L-module to belong to the principal block which slightly generalizes **[Fe4**, Proposition 3**]**:

Proposition 2. *Let L be a finite-dimensional restricted Lie algebra, and let S be a simple restricted L-module. If S belongs to the principal block of L, then* $\operatorname{Nil}(L) \subseteq \operatorname{Ann}_L(S)$.

Proof. Let x be an arbitrary element in $\operatorname{Nil}(L)$. According to **[SF**, Theorem 2.3.4**]**, there exists a positive integer n such that $x^{[p]^n}$ is semisimple. Hence we obtain from **[Fe4**, Proposition 3**]** that

$$x^{[p]^n} \in T_p(\operatorname{Nil}(L)) = T_p(L) \subseteq \operatorname{Ann}_L(S).$$

Since S is restricted, $\mathrm{Nil}(L)$ acts nilpotently on S, and by virtue of the Engel-Jacobson theorem, $\mathrm{Nil}(L)$ acts trivially on S. □

Remark. It is an immediate consequence of Proposition 2 that the principal block of a finite-dimensional nilpotent restricted Lie algebra only contains the one-dimensional trivial module (see [**Voigt**, Satz 2.29], [**Pet**], the proof of [**Fe3**, Theorem 1], and [**Fe4**, Theorem 2]).

In particular, if L is solvable, Proposition 2 in conjunction with [**Fe4**, Proposition 2] and Proposition 1 yields the following special case of the analogue of a result for finite-dimensional modular group algebras due to Brauer (see [**Mich**, Theorem 2]).

Corollary 2. *Let L be a finite-dimensional solvable restricted Lie algebra. Then*

$$\mathrm{Nil}(L) = \bigcap_{S \in \mathbb{B}_0(L)} \mathrm{Ann}_L(S). \qquad \square$$

Now we are ready to prove the following analogue of [**Will**, Theorem 1.8] (see also [**FG**, Theorem 2.6]) for simple modules in the principal block of a restricted Lie algebra:

Theorem 2. *Let L be a finite-dimensional restricted Lie algebra. Then every simple restricted L-module belonging to the principal block of L is a submodule of a suitable tensor product of (finitely many) chief factors of L.*

Proof. Let S be a simple restricted L-module which belongs to the principal block of L. By Proposition 2, S is a simple restricted $L/\mathrm{Nil}(L)$-module. Since $u(L/\mathrm{Nil}(L))$ is a Frobenius algebra (cf. [**SF**, Corollary 5.4.3]), Ikeda's theorem (cf. [**CR**, Theorem 62.11]) implies that the injective hull of S (with respect to $u(L/\mathrm{Nil}(L))$) is a projective indecomposable $u(L/\mathrm{Nil}(L))$-module. As S can be embedded into its injective hull, the assertion is now an immediate consequence of Theorem 1. □

Remark. Note that Theorem 2 can also be derived directly from *Burnside's theorem* for restricted Lie algebras (see [**FK**, Corollary 3.4]) which is a consequence of [**PQ**, Corollary 10] and the discussion preceding [**PQ**, Theorem 4] but not (explicitly) contained in [**PQ**].

A chief factor I/J of L is called *abelian* if I/J is abelian (as a Lie algebra), i.e., $[I, I] \subseteq J$. The following example shows that for non-solvable restricted Lie algebras it is *not* enough, in Theorem 2, to consider only tensor products of *abelian* chief factors.

Example 1. Consider $L := \mathfrak{gl}_2(\mathbb{F})$ over an algebraically closed field $\mathbb{F}$ of characteristic $p > 2$. Then the trivial module is the only abelian chief factor of L but the $(p-1)$-dimensional restricted $\mathfrak{sl}_2(\mathbb{F})$-module with trivial central action belongs to the principal block of L.

In order to illustrate Theorem 2, we want to apply it to an important special case. A Lie algebra L is called *supersolvable* if every chief factor of L is one-dimensional. Hence Theorem 2 implies that every simple module in the principal block of a supersolvable restricted Lie algebra is one-dimensional (see also [**Fe2**, Theorem 2] or [**Fe4**, Theorem 1] for a cohomological proof). In this case the principal block $\mathbb{B}_0(L)$ forms a commutative group under

$$[S_1] + [S_2] := [S_1 \otimes_{\mathbb{F}} S_2] \quad \forall\, [S_1], [S_2] \in \mathbb{B}_0(L),$$

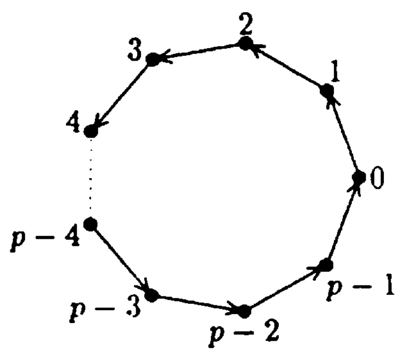
with the isomorphism class $[\mathbb{F}]$ of the one-dimensional trivial module as the identity element and the isomorphism class $[S^*]$ of the dual module as the inverse element of $[S]$. Note that via $\gamma \mapsto [F_\gamma]$ the principal block $\mathbb{B}_0(L)$ is isomorphic to the group G_0^L introduced in **[Fe4]**. Then Theorem 2 immediately yields the following generalization of **[Fe5**, Lemma 4.9**]**.

Corollary 3. *Let L be a finite-dimensional supersolvable restricted Lie algebra. Then the principal block of L is generated by the (non-trivial) chief factors of L.* □

Example 2. Consider the non-abelian two-dimensional restricted Lie algebra

$$L := \mathbb{F}t \oplus \mathbb{F}e, \quad [t, e] = e, \quad t^{[p]} = t, \quad e^{[p]} = 0.$$

Then the restricted universal enveloping algebra of L has the following Gabriel quiver



where the vertices corresponding to the one-dimensional restricted L-modules F_τ are labelled by the respective eigenvalues τ of t (see **[Fe1**, Beispiel II.4.1**]**). Hence the principal block $\mathbb{B}_0(L)$ of L is generated by the only non-trivial (split) chief factor F_1, i.e., in this case $\mathbb{B}_0(L)$ is a cyclic group of order p.

Recall that a chief factor I/J of L is called *split* if the exact sequence $0 \to I/J \to L/J \to L/I \to 0$ splits (as a sequence of Lie algebras). It is clear that the top chief factor is *always* split. If in addition L is *unimodular*, the proof of **[Fe5**, Lemma 4.9**]** shows that then $\mathbb{B}_0(L)$ is even generated by the (non-trivial) *split* chief factors of L. Note that the same proof (using **[Sta**, Corollary 2**]** and **[Ba2**, Theorem 1**]** instead of **[Fe4**, Theorem 1**]** and **[Ba1**, Theorem 1**]**) shows that the principal block of a finite p-supersolvable group is *always* generated by its (non-trivial) split chief factors (because the group algebra is *always* symmetric). As a generalization of Example 2 we conclude this note by the following sufficient condition (partly complementary to **[Fe5**, Lemma 4.9**]**) pointing towards this stronger result in the case of restricted Lie algebras:

Proposition 3. *Let L be a finite-dimensional supersolvable restricted Lie algebra. If* $\mathrm{Nil}(L)$ *is abelian, then every chief factor of L is split.*

Proof. Assume first that the ground field $\mathbb{F}$ is algebraically closed. Since L is supersolvable, there exists a torus T such that $L = T \oplus \mathrm{Nil}(L)$ (see the proof of [**Fe4**, Theorem 4]). Let S be a non-trivial chief factor of L. According to the main result of [**Ho**], the adjoint module L is a semisimple T-module. Since S is not trivial, this in conjunction with the five-term exact sequence for Lie algebra cohomology and Proposition 1 yields

$$\dim_{\mathbb{F}} H^1(L, S) = \dim_{\mathbb{F}} \mathrm{Hom}_T(\mathrm{Nil}(L), S) = \dim_{\mathbb{F}} \mathrm{Hom}_T(L, S) = [L : S] \neq 0,$$

where $[L : S]$ denotes the multiplicity of S as a composition factor of the adjoint module L. If S is trivial, then we obtain

$$H^1(L, S) \cong L/[L, L] \neq 0$$

because $L (\neq 0)$ is solvable. Hence in both cases [**Ba1**, Theorem 1] shows that S is split.

If $\mathbb{F}$ is arbitrary, then let $\overline{\mathbb{F}}$ denote the algebraic closure of $\mathbb{F}$. If S is a chief factor of L, then $S \otimes_{\mathbb{F}} \overline{\mathbb{F}}$ is a direct sum of chief factors of $L \otimes_{\mathbb{F}} \overline{\mathbb{F}}$. Since $L \otimes_{\mathbb{F}} \overline{\mathbb{F}}$ is also supersolvable with $\mathrm{Nil}(L \otimes_{\mathbb{F}} \overline{\mathbb{F}}) = \mathrm{Nil}(L) \otimes_{\mathbb{F}} \overline{\mathbb{F}}$ abelian, we can conclude from the first part of the proof that every direct summand of $S \otimes_{\mathbb{F}} \overline{\mathbb{F}}$ is split. But then

$$H^1(L \otimes_{\mathbb{F}} \overline{\mathbb{F}}, S \otimes_{\mathbb{F}} \overline{\mathbb{F}}) \cong H^1(L, S) \otimes_{\mathbb{F}} \overline{\mathbb{F}}$$

in conjunction with a twofold application of [**Ba1**, Theorem 1] implies that S is split. □

It is easy to see that in general *not* every chief factor of a supersolvable restricted Lie algebra is split, but in the numerous examples (with $\dim_{\mathbb{F}} \mathrm{Nil}(L) \leq 7$) computed by the author it is always true that

$$\mathrm{Nil}(L) = \bigcap_{S \in \mathfrak{F}_{\mathrm{split}}(L)} \mathrm{Ann}_L(S),$$

where $\mathfrak{F}_{\mathrm{split}}(L)$ denotes the set of isomorphism classes of split chief factors of L. The author was not able to decide whether this slightly stronger version of Proposition 1 does hold (at least) for a finite-dimensional supersolvable restricted Lie algebra, and consequently whether the precise analogue of the group-theoretic result as mentioned before Proposition 3 would or would not be valid.

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